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Path planning using Harmonic Functions and Probabilistic Cell Decomposition*

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***Abstract*— Potential-field approaches based on harmonic functions have good path planning properties, although the explicit knowledge of the robot's Configuration Space is required. To overcome this drawback, a combination with a random sampling scheme is proposed. Harmonic functions are computed over a 2^d-tree decomposition of a d-dimensional Configuration Space that is obtained with a probabilistic cell decomposition (sampling and classification). Cell sampling is biased towards the more promising regions by using the harmonic function values. Cell classification is performed by evaluating a set of configurations of the cell obtained with a deterministic sampling sequence that provides a good uniform and incremental coverage of the cell. The proposed planning framework opens the use of harmonic functions to higher dimensional $\mathcal{C}$-spaces.**

***Index Terms*— Path planning, harmonic functions, random sampling, deterministic sampling.**

I. INTRODUCTION

The planning of collision-free paths for a robot through the obstacles in a workspace is a difficult problem that is usually tackled in the robot's Configuration Space ($\mathcal{C}$-space). In $\mathcal{C}$-space the robot is mapped to a point and the obstacles in the workspace are enlarged accordingly ($\mathcal{C}$-obstacles). The $\mathcal{C}$-space explicitly captures the motion freedom of the robot, which facilitates path planning issues. Nevertheless, the dimension of the $\mathcal{C}$-space is generally high, since it is equal to the number of degrees of freedom of the robot, and the characterization of $\mathcal{C}$-obstacles is a barrier that precludes the use of many motion planning approaches.

Sampling-based motion planners, like Probabilistic Roadmap Methods (PRM [1]) or those based on the Rapidly-exploring Random Trees (RRT [2]), are giving very good results in robot path planning problems with many degrees of freedom. Its success is mainly due to the fact that they are sampling-based, i.e. the explicit characterization of $\mathcal{C}$-obstacles is not required and only sample configurations of $\mathcal{C}$-space are checked for collision.

To improve sampling efficiency and to be able to find a path with as few $\mathcal{C}$-space samples as possible, several variants have been proposed to bias the sampling towards the most promising and difficult regions. For instance, a sample distribution is defined such that it increases the number of samples on the border of the $\mathcal{C}$-obstacles [3], or around the medial axis of the free $\mathcal{C}$-space [4], or around the initial and goal configurations [5]. The use of an artificial potential field is also proposed to bias the sampling towards narrow passages [6] [7]. In comparison to those approaches, sampling efficiency can also be achieved by deterministic sampling sequences [8] [9]. Deterministic sampling sequences have the advantages of classical grid search approaches (i.e. a lattice structure) and a good uniform coverage of the $\mathcal{C}$-space. These features are the reasons why they have given very good results in PRM-like planners [10].

Other path planning approaches, like those based on potential-field methods have also given good results, although not in high dimensional $\mathcal{C}$-spaces, since usually an approximate decomposition of $\mathcal{C}$-space is required [11]. Among them, those based on harmonic functions are interesting because they give rise to practical, resolution-complete planners without local minima [12]. Some attempts have been carried to extend the use of potential-field based methods to higher dimensional $\mathcal{C}$-spaces, by combining them with a random sampling scheme (in a similar way as roadmap methods gave rise to PRM). For instance, Yang and LaValle [13] propose a global navigation function defined over a collection of spherical balls of different radius that cover the free $\mathcal{C}$-space. Those balls are arranged as a graph that is incrementally build following sampling-based techniques. The approach we propose is called Probabilistic Harmonic-function-based Method (PHM [14]) and combines harmonic functions and random sampling. PHM has the following main features: i) it allows the use of the harmonic functions approach without the explicit knowledge of the $\mathcal{C}$-space, because a 2^d-tree decomposition of a d-dimesional $\mathcal{C}$-space is partially explored by the random sampling of cells and their classification using a distance checker, and ii) the random sampling of cells is biased by the harmonic function values, which allows to find a solution channel having explored as few cells as possible.

This paper further extends the PHM approach by introducing a classification method for the sampled cells that does not rely on the availability of a distance checker. This method classifies cells as free, obstacle or partially free depending on the collision test evaluated at a set of configurations of the cell (a similar method is proposed by Lingelbach [15]). A deterministic sampling sequence is

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proposed to generate those test configurations. The degree of knowledge of the type of cell is then used to introduce another bias to the random cell sampling.

The paper is structured as follows. Section II reviews the basic PHM path planning approach. Sections III and IV introduce, respectively, the cell classification and the cell sampling procedures. Section V presents the general planning algorithm that is illustrated with an example. Finally Section VI summarizes the contributions.

II. Basic PHM approach

A. $\mathcal{C}$-space decomposition

The following 2^d-tree decomposition of a d-dimensional $\mathcal{C}$-space is considered [16]. An initial cell, b^0, covering the entire $\mathcal{C}$-space is the tree root (b^0 is considered to have sides with unitary size). The levels in the tree are called partition levels and are enumerated such that the tree root is the partition level 0 and the maximum resolution corresponds to partition level M. Partition levels are denoted by super-indices: a cell of a given partition level m is called an m-cell, and is denoted as b^m. The m-cells have sides of size $s_m = 1/2^m$.

A code convention that univocally labels and locates each cell of the 2^d-tree decomposition of $\mathcal{C}$-space is used. With this code convention, any subset of cells, e.g. those belonging to the subset $\mathcal{C}_{free}$ of collision-free configurations, can be managed as a list of codes, in a similar way as the *linear quadtrees* proposed in [17] for $d = 2$.

The cell codes are non-negative integers that univocally locate the cells in $\mathcal{C}$-space. The codes for a given partition level m range from C^m_{ini} to C^m_{end}, with:

$$C^m_{ini} = \frac{2^{dm} - 1}{2^d - 1} \tag{1}$$

$$C^m_{end} = 2^d C_{ini} \tag{2}$$

Since $C^m_{ini} = C^{(m-1)}_{end} + 1$, the proposed code convention uses all non-negative integers. Fig. 1 shows the codes used for the cells of different partition levels for 2D and 3D $\mathcal{C}$-spaces.

Let $V^m_{b_k} = (v^m_1, \ldots, v^m_d)$ be the coordinates of an m-cell, b^m_k, with each component expressed in base 2 as $v^m_j = (a_{mj} \ldots a_{1j})_2$. Then, the code k of the cell is computed using the following expression:

$$k = C^m_{ini} + \sum_{i=1}^{m} r_i 2^{d(i-1)} \tag{3}$$

where $r_i = (a_{id} \ldots a_{i1})_2 \;\; \forall i \in 1 \ldots m$ are the coefficients retrieved from the following table:

Code	v^m_d	$\ldots$	v^m_j	$\ldots$	v^m_1
r_m	a_{md}	$\ldots$	a_{mj}	$\ldots$	a_{m1}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
r_i	a_{id}	$\ldots$	a_{ij}	$\ldots$	a_{i1}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
r_1	a_{1d}	$\ldots$	a_{1j}	$\ldots$	a_{11}

Note that from (3), it can be seen that the values $r_i \;\; \forall i \in 1 \ldots m$ are the digits of the representation in base 2^d of $(k - C^m_{ini})$. The level of a cell with code k can be obtained as follows:

$$m = (\text{int}) \left\{ \frac{1}{d} \log_2[(2^d - 1)k + 1] \right\} \tag{4}$$

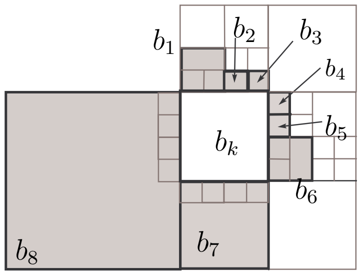

Fig. 2. Neighbor cells of different size used to compute the harmonic function at b_k. Cells b_2, b_3, b_4 and b_5 are M-cells.

B. Harmonic Functions

An harmonic function ϕ on a domain $\Omega \subset \Re^n$ is a function that satisfies Laplace's equation [12]:

$$\nabla^2 \phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{5}$$

Numerical solutions to Laplace's equation can be found using finite difference methods by sampling ϕ and its derivatives on a regular grid and using relaxation methods. Solutions over a non-regular grid like a 2^d-tree decomposition are also possible (they are faster obtained and the only influence in the planning strategy is that the solution channels obtained are composed of cells of different size). Let b_k be a given cell with code k, N be the actual number of neighbors of b_k, and W_j be the number of M-cells that can be contained in the border between b_k and a neighbor cell b_j. Then, the value u_k of the harmonic function, called the HF value, at b_k is computed as the following weighted mean [16]:

$$u_k = \frac{\sum_{j=1}^{N} W_j u_j}{N_{max}} \tag{6}$$

As an example, in Fig. 2 $N = 8$, $N_{max} = 16$ and:

$$u_k = \frac{2u_1 + u_2 + u_3 + u_4 + u_5 + 2u_6 + 4u_7 + 4u_8}{16} \tag{7}$$

Relaxation methods are applied all over the cells of the 2^d-tree in a hierarchical top-down manner, i.e. the HF values are consecutively computed from level 0 to level M in an iterative way until the convergence is attained. The obtained solution depends on the boundary conditions. The Dirichlet boundary condition is used, which fixes the cells of the $\mathcal{C}$-obstacles' border at a high value and the goal cell at a low one.

C. PHM procedure

Let:

G: be the set of partially free cells (gray).
B: be the set of obstacle cells (black).
W: be the set of free cells (white).

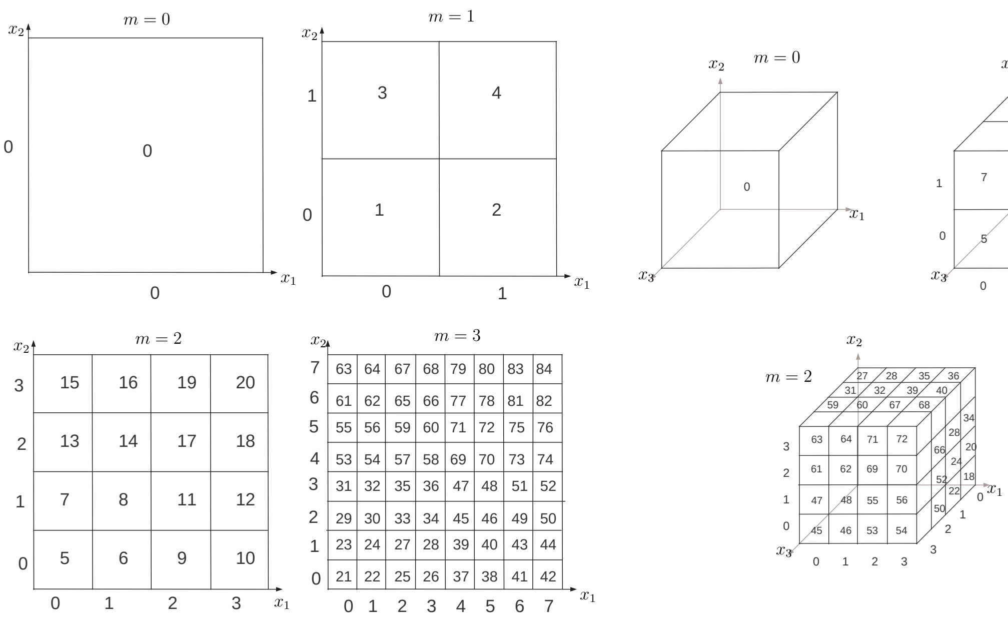

Fig. 1. Cell codes for different levels in the hierarchy in 2D and 3D $\mathcal{C}$-spaces.

The basic PHM path planning algorithm [14] consecutively performs the following steps until either a path is found or the non-existence of a path at the maximum resolution allowed is reported:

1) Explore: Randomly samples from G a predefined number of cells and, depending on the cell size and the signed distance[1] from the center of the cell to the obstacles, classifies them into the sets W, B or G (in this latter case the cell is partitioned and the subcells are returned to G). Cell sampling is biased by:
   a) The cell size: Sampling probability is set proportional to the cell size. In this way the uncertainty of big partially free cells is elucidated earlier.
   b) The HF values: Sampling probability is set inversely proportional to the HF values, that are available once step 2 (Compute-HF) has been executed a first time. In this way, sampling is focused around the goal cell, since cells near the goal have a lower HF value. Moreover, cells with a higher HF value than the initial cell are discarded, i.e. its sampling probability is set to zero, since they will never appear in a solution path that is found following a gradient descent.
2) Compute-HF: Computes the harmonic function over the white and gray cells.
3) Find-path: Obtains the solution path (a channel of free and/or partially free cells) from the cell containing the initial configuration to the cell containing the goal one by iteratively selecting the neighbor cell with lowest HF value. A solution channel is found if it exists, since harmonic functions do not have local minima. When the solution channel have partially free cells, they are checked to be free cells. When this is the case the procedure exits with the solution channel found, otherwise it returns to step 1.

[1]A negative distance corresponds to a penetrating distance.

III. Cell classification

In the basic PHM approach cell classification relies on the availability of a distance checker able to provide positive as well as negative (penetrating) distances. This requirement is relaxed in the present paper by performing a probabilistic cell classification based on the collision checking (without distance information) at a set of configurations of the cell obtained with a deterministic sampling sequence.

A. Deterministic sampling sequence

To uniformly sample a given subspace (a cell) it is more efficient to use a deterministic sampling sequence than doing it randomly, since deterministic sampling sequences provide a better uniform and incremental space coverage [9].

Let L_d be a low-dispersion ordering of the 2^d descendant cells of a given parent cell. If the position of a cell with respect to its parent cell is defined by a binary word with d bits, one for each axis, then L_d can be found as the sequence of 2^d binary words such that each element of the sequence maximizes the distance[2] to the previous elements of the sequence [18]. For instance, for two and three dimensional $\mathcal{C}$-spaces the ordering can be $L_2 = \{00, 11, 01, 10\}$ and $L_3 = \{000, 111, 010, 101, 001, 110, 011, 100\}$, respectively. Alternative orderings satisfying the same feature are also possible.

[2]The distance between two binary numbers is measured as the number of bits that differ, and is equivalent to the Manhattan distance between the cells they represent.

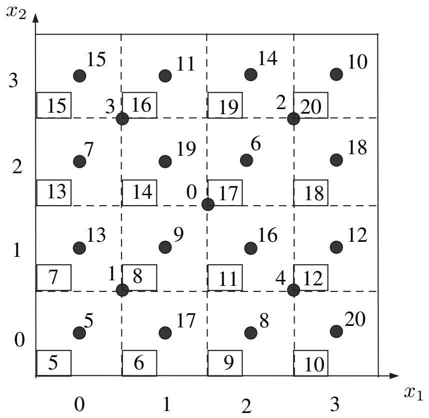

Fig. 3. Sampled configurations of a 2D $\mathcal{C}$-space located at the center of the sampled cells. The configuration labels indicate the index in the generation sequence. Boxed numbers are the codes of the 2-cells.

The deterministic sampling sequence, called $s_d(k)$, is based on applying recursively L_d. Given an m-cell with code K, b_K^m, the sequence $s_d(k)$ provides an ordering of codes of cells whose centers are the samples that uniformly and incrementally cover b_K^m.

Let $k \geq 0$ be the index of the sequence, r_i be the digits of $(k - C_{ini}^m)$ in base 2^d as expressed in (3), and m be the hierarchical level associated to k as expressed in (4). Then:

$$s_d(k) = K2^{dm} + C_{ini}^m + \sum_{i=1}^{m} L_d(r_i) 2^{d(m-i-1)} \quad (8)$$

As an example, the first cell samples generated over the cell b^0 that contains the whole 2D $\mathcal{C}$-space and whose label is $K = 0$ are (Fig. 3):

$$\begin{aligned} s_2(k) \;=\; & \{0, 1, 4, 3, 2, 5, 17, 13, 9, 8, 20, 16, 12, \\ & 7, 19, 15, 11, 6, 18, 14, 10, \dots\} \end{aligned} \quad (9)$$

And the samples generated over the cell with label $K = 4$ are:

$$\begin{aligned} s_2(k) \;=\; & \{4, 17, 20, 19, 18, 69, 81, 77, 73, 72, 84, 80, 76, \\ & 71, 83, 79, 75, 70, 82, 78, 74, \cdots\} \end{aligned} \quad (10)$$

B. Probabilistic cell classification

A given m-cell is classified depending on the result of the collision check performed at the configuration of its center and at the configurations of the centers of its subcells. The configurations of this subset are consecutively obtained by the sampling sequence $s_d(k)$. The maximum number of configurations generated by $s_d(k)$ to classify the cell is:

$$J = \sum_{i=m}^{M} 2^{d(i-m)} \quad (11)$$

that corresponds to the cardinality of the set composed of the given m-cell and all of its subcells, of levels ranging from $m + 1$ to M. Nevertheless, cell classification is first performed having generated and evaluated only $j < J$ configurations, and is then updated each time new samples are provided.

Assume that j configurations have been generated and checked as collision-free configurations. Then, at this point, the cell can be classified as a free cell with a certain degree of certainty, called color certainty, defined as:

$$C_{color} = \frac{j}{J} \quad (12)$$

An analogous classification can be stated if all of the first j configurations are checked as obstacle configurations. As an example Fig. 4a shows a cell classified as a free cell (W) with $C_{color} = 17/21$, and Fig. 4b shows a cell classified as an obstacle cell (B) with $C_{color} = 8/21$.

When all of the J configurations are checked to be either collision-free or obstacle configurations, the color of the cell is known with certainty (up to the maximum resolution level), and $C_{color} = 1$. Cells not yet explored are grouped in set G and have $C_{color} = 0$.

When any two of those test configurations have a different collision-check result, the cell cannot be classified as either free or obstacle, since it contains both free and obstacle configurations. In this case the cell is subdivided as necessary until the obtained subcells contain only free configurations, or obstacle configurations, or do not contain any sample at all. In the first two cases these subcells are classified as free or obstacle cells (with their own degree of certainty), and those void subcells are classified as unknown subcells. As an example Fig. 4c shows a cell where the first seven samples are classified as obstacle but the eighth (sample number 7) is classified as free. The cell is partitioned into: a white M-cell with $C_{color} = 1$, two black cells with $C_{color} = 2/5$, one black cell with $C_{color} = 1/5$, and three gray M-cells.

IV. Cell Sampling

Cell sampling is performed in order to explore the unknown $\mathcal{C}$-space and find a solution channel through free cells from an initial cell b_{ini} to a goal cell b_{goal}. Cell sampling must be carefully done in order to be able to find a solution having explored as few cells as possible.

With this aim in mind, the probability to sample a cell b_k is determined by the weight ω_{b_k}:

$$\omega_{b_k} = (\omega_{1_{b_k}})^{a_1} (\omega_{2_{b_k}})^{a_2} (\omega_{3_{b_k}})^{a_3} \quad (13)$$

This weight considers the following three items:

- The probability to sample a cell is set increasing with its size. If b_k is an m-cell then:

$$\omega_{1_{b_k}} = 2^{-m} \quad (14)$$

- Cells near b_{goal} are set at a higher sampling probability, and those cells that do not have any chance to be on the solution channel are discarded for sampling. If H_{WG} is the harmonic function computed over the white and gray cells, then:

$$\omega_{2_{b_k}} = \begin{cases} 0 & \text{if } H_{WG}(b_k) > H_{WG}(b_{ini}) \\ \frac{H_{WG}(b_{ini}) - H_{WG}(b_k)}{H_{WG}(b_{ini}) - H_{WG}(b_{goal})} & \text{otherwise} \end{cases} \quad (15)$$

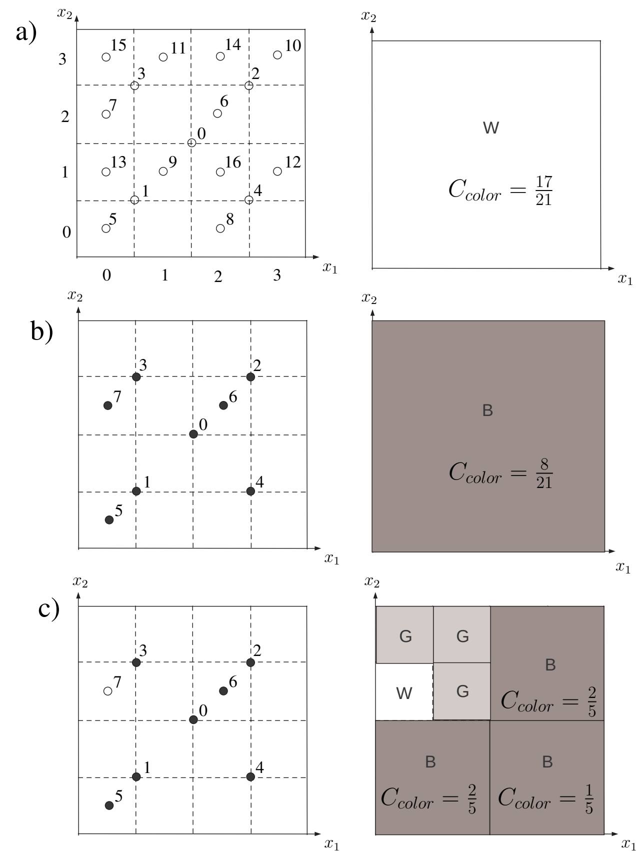

Fig. 4. Cell classification: a) free cell b) obstacle cell c) cell subdivided into free, obstacle and unknown subcells.

- The probability to sample a cell is set increasing with its color uncertainty:

$$\omega_{3_{b_k}} = 1 - C_{color} \tag{16}$$

  Gray cells have $\omega_{3_{b_k}} = 1$ and cells whose color certainty is $C_{color} = 1$ have $\omega_{3_{b_k}} = 0$ and therefore are not eligible for sampling.

The coefficients a_i in (13) are currently set to $a_1 = 1$, $a_2 = 1$ and $a_3 = 1$. A thorough analysis is required to evaluate their influence in different $\mathcal{C}$-spaces and select the best values in each case.

Finally, the sampling procedure is performed by using a cumulative function, $F(b_k)$, defined as follows over the cells (Fig. 5):

$$F(b_k) = \sum_{\forall \omega_{b_i} < \omega_{b_k}} \omega_{b_i} \tag{17}$$

V. Planning procedure

The planning procedure iteratively explores the $\mathcal{C}$-space and tries to find a path using the guidance of harmonic functions. It has the following steps:

1) Sample a set of cells of $\mathcal{C}$-space following the procedure detailed in Section IV.
2) Classify the sampled cells following the procedure detailed in Section III. For each cell, according to the results, either update its color certainty or partition the cell into subcells as necessary.
3) Compute the harmonic function over the white and gray cells, fixing the black cells at a high value and the goal cell at a low one.
4) Find a channel of cells from b_{ini} to b_{goal} following the gradient descent. If no channel exists go back to step 1.
5) The solution channel found is composed of white cells (each one with its color certainty) and/or gray cells (totally unexplored). Then:
   a) Subdivide the channel cells into M-cells.
   b) Check for collision at the configuration of their centers and classify them as either white or black.
   c) Compute a new harmonic function over the white ones.
   d) Search a path. If it exists, return the solution path. Otherwise, go back to step 1.

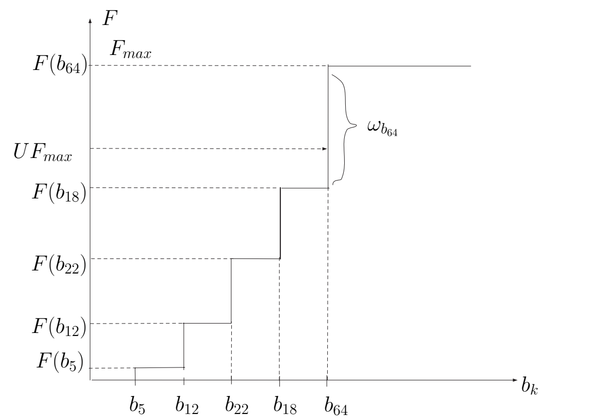

Fig. 5. Random sampling using cumulative function $F(b_k)$. A randomly chosen value $U \in [0, 1]$ determined that cell b_{64} be chosen.

As an example, Fig.s 6 shows a 2D $\mathcal{C}$-space at different steps of the path planning algorithm. Fig. 6a shows the initial non-regular grid composed of gray cells covering the whole $\mathcal{C}$-space except b_{ini} and b_{goal}. Fig. 6b shows the $\mathcal{C}$-space after the first cell sampling and classification. Fig. 6c shows the obtained solution channel from b_{ini} to b_{goal} (red cells) and the partially free cells that were discarded for exploration (blue cells). Fig. 6d shows the solution path composed of M-cells obtained by refining the solution channel. In this example, approximately a 20% of the 4096 M-cells of the $\mathcal{C}$-space have been tested for collision.

VI. Summary

Sampling-based methods are giving very good results in robot path planning problems with many degrees of freedom. On the other hand, potential-field methods based on harmonic functions have interesting path planning properties. In this paper the combined use of harmonic functions and a sampling-based scheme is proposed. Harmonic functions are computed over a non-regular grid

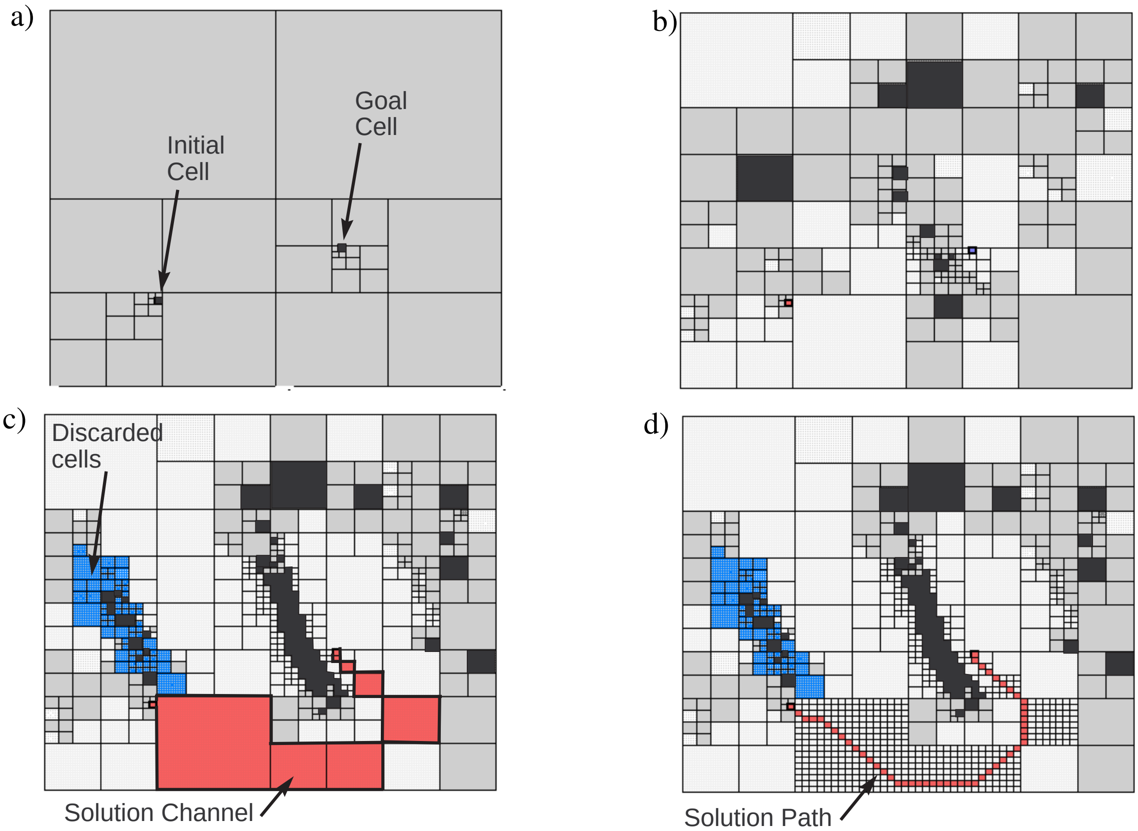

Fig. 6. a) Initial non-regular grid, b) $\mathcal{C}$-space after the first cell sampling and classification c) Solution channel d) Solution path of M-cells.

decomposition of a d-dimensional $\mathcal{C}$-space that is obtained with the probabilistic sampling of cells. Sampled cells are classified as free, obstacle or partially free cells by evaluating a set of configurations of the cell obtained with a deterministic sampling sequence that provides a good uniform and incremental coverage of the cell. Cell sampling is biased towards the more promising regions by using the harmonic function values, the cell size and the degree of knowledge of the type of cell. The combination of harmonic functions and a probabilistic cell decomposition opens the use of harmonic functions to higher dimensional $\mathcal{C}$-spaces.

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